CONCISE MATHEMATICAL MODELING (MATH311) WITH COMPREHENSIVE SOLUTION METHODOLOGY

COMPILED AND SOLVED

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Ordinary Differential Equation

$$P(x)y'' + Q(x)y' + R(x)y = G(x)$$

When $G(x) = 0$, we have a Homogeneous Linear Equation

If $G(x) \neq 0$, we have Non-Homogeneous Equation

And when P, Q and R are constant functions, we have the following equation.

$$ay'' + by' + cy = 0$$

Now let $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2e^{mx}$, and substitute in the equation above

ie

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$e^{mx}[am^2 + bm + c] = 0$$

$$e^{mx} \neq 0$$

$$am^2 + bm + c = 0$$

Now when we have the above equation, we solve for the value of m. And for that it is either we use factorization method or we use General formular method which is given as follows;

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three cases you need to know when solving for m.

CASE I

When

$b^2 - 4ac > 0$; we are going to have two real and unequal roots

e.g. $m_1 = 4$ $m_2 = -3$

Now when we have such situation the General solution to the Differential Equation will be

$$\boxed{y = C_1e^{m_1x} + C_2e^{m_2x}}$$

CASE II

When

$b^2 - 4ac = 0$, then we have real and equal roots, e.g. $m = 5$

The General solution to such related problem is

$$\boxed{Y = C_1e^{mx} + C_2xe^{mx}}$$

CASE III

When

$b^2 - 4ac < 0$; in this case we get imaginery or complex numbers, so you have to use the Quadratic formula. Example of such complex numbers is

$m_1 = \alpha + \beta i \quad m_2 = \alpha - \beta i$

The General solution to such problem is

$$\boxed{Y = e^{\alpha x}\left[C_1 \cos(\beta x) + C_2 \sin(\beta x)\right]}$$

Examples

1) $y'' - 5y' + 6y = 0$

let $y = e^{mx}, y' = me^{mx}, y'' = m^2e^{mx}$

$m^2e^{mx} - 5me^{mx} + 6e^{mx} = 0$

$e^{mx}\left[m^2 - 5m + 6\right] = 0$

$e^{mx} \neq 0$

$m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$m(m-2) - 3(m-2) = 0$

$(m-2)(m-3) = 0$

$(m-2) = 0 \quad (m-3) = 0$

$m_1 = 2 \quad m_2 = 3$

Since the roots are real and unequal, we therefore use General solution of CASE I

i.e

$y = C_1e^{m_1x} + C_2e^{m_2x}$

but $m_1 = 2$ and $m_2 = 3$

$$\boxed{\therefore y = C_1e^{2x} + C_2e^{3x}}$$

2) $y'' - 6y' + 9 = 0$

$am^2 + bm + c = 0$

$m^2 - 6m + 9 = 0$

$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$(m-3)(m-3) = 0$

$m-3 = 0 \quad m-3 = 0$

$m = 3$

Since the roots are real and equal, then the General solution of case II will be used...

i.e $y = C_1e^{mx} + C_2xe^{mx}$

$$\boxed{\therefore y = C_1e^{3x} + C_2xe^{3x}}$$

(3) $9\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 16y = 0$

$9y'' + 24y' + 16y = 0$

$am^2 + bm + c = 0$

$9m^2 + 24m + 16 = 0$

$9m^2 + 12m + 12m + 16 = 0$

$3m(3m+4) + 4(3m+4) = 0$

$(3m+4)(3m+4) = 0$

$(3m+4)^2 = 0$

$3m + 4 = 0$

$\frac{\cancel{3}m}{\cancel{3}} = \frac{-4}{3}$

$m = -4/3$

$$\boxed{\therefore y = c_1 e^{-\frac{4}{3}x} + c_2 x e^{-\frac{4}{3}x}}$$

(4) $y'' + 8y' + 25y = 0$

$am^2 + bm + c = 0$

$1m^2 + 8m + 25 = 0$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \frac{-8 \pm \sqrt{8^2 - 4(1)(25)}}{2(1)}$

$= \frac{-8 \pm \sqrt{64 - 100}}{2}$

$= \frac{-8 \pm \sqrt{-36}}{2}$

$= \frac{-8 \pm 6i}{2}$

$= -4 \pm 3i$

$m_1 = -4 + 3i \quad m_2 = -4 - 3i$

$m_1 = \alpha + \beta i \quad m_2 = \alpha - \beta i$

$\alpha = -4 \quad \beta = 3$

Here since we have complex number, then the General solution of CASE III will be used.

i.e

$y = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

$$\boxed{\therefore y = e^{-4x}[c_1 \cos(3x) + c_2 \sin(3x)]}$$

Initial Value Problem (IVP)

(5) $y'' + 4y = 0 \quad y(0) = 4, y'(0) = 6$

$y'' + 0y' + 4y = 0$

$am^2 + bm + c = 0$

$1m^2 + 0m + 4 = 0$

$m^2 + 4 = 0$

$\sqrt{m^2} = \sqrt{-4}$

$m = \pm 2i$

$m_1 = 0 + 2i \qquad m_2 = 0 - 2i$

$m_1 = \alpha + \beta i \qquad m_2 = \alpha - \beta i$

$$y = e^{\alpha x}\left[C_1\cos(\beta x) + C_2\sin(\beta x)\right]$$

$$y = e^{0\times x}\left[C_1\cos(2x) + C_2\sin(2x)\right]$$

$$y = C_1\cos(2x) + C_2\sin(2x)$$

Recall that $y(0)=4$ & $y'(0)=6$

Now,

$$4 = C_1\cos(2\times0) + C_2\sin(2\times0)$$

$$4 = C_1(1) + C_2(0)$$

$$4 = C_1$$

Also

$$y' = -2C_1\sin(2x) + 2C_2\cos(2x)$$

$$6 = -2C_1\sin(2\times0) + 2C_2\cos(2\times0)$$

$$6 = -\cancel{2C_1(0)}^{\,0} + 2C_2(1)$$

$$\frac{6}{2} = \frac{2C_2}{2}$$

$$3 = C_2$$

Now Substitute the values of C_1 and C_2 in the General solution of the Differential Equation i.e

$$\boxed{\therefore y = 4\cos(2x) + 3\sin(2x)}$$

Boundry Value Problem (BVP)

(6) $y'' - 2y' + y = 0 \quad y(0)=3,\ y(1)=7e$

$$am^2 + bm + c = 0$$

$$m^2 - 2m + 1 = 0$$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)(m-1) = 0$$

$$(m-1)^2 = 0$$

$$m = 1$$

$$y = C_1e^{mx} + C_2xe^{mx}$$

$$y = C_1e^{x} + C_2xe^{x}$$

$$3 = C_1e^0 + C_2(0)e^0$$

$$3 = C_1$$

Also for $y(1)$, we say;

$$7e = 3e^1 + C_2(1)e^1$$

$$7e = 3e + C_2e$$

$$7e - 3e = C_2e$$

$$4\cancel{e} = C_2\cancel{e}$$

$$4 = C_2$$

$$\boxed{\therefore y = 3e^x + 4xe^x}$$

Exercises

1) Find the general solution of each ~~of~~ equation.

i) $y'' - 9y = 0$

ii) $y'' + 4y' - 5y = 0$

iii) $6y'' + y' - y = 0$

2) Solve each Initial Value Problem

i) $y'' + y' - 12y = 0, \quad y(0) = -2, y'(0) = -20$

ii) $y'' + y' - 12y = 0, \quad y(\pi) = -2, y'(\pi) = -20$

iii) $y'' + 2y' - 3y = 0, \quad y(0) = 1, y'(0) = 13$

iv) $y'' + 2y' - 4y = 0, \quad y(0) = 6, y'(0) = -6$

v) $y'' + 2y' - 4y = 0, \quad y(18) = 6, y'(18) = -6$

Method of Undetermined Coefficient

The method of undetermined coefficients can be used to find the general solution to:

$$y'' + p(x)y' + q(x)y = g(x) \quad --(1)$$

General solution:

$$\boxed{y(x) = c_1 y_1 + c_2 y_2 + Y_p(x)}$$

STEPS

1) Solve the corresponding homogeneous DE.

$$y'' + p(x)y' + q(x)y = 0$$

2) Guess the form of a particular solution to (1) with an undetermined coefficients.

3) Perform substitution into (1) and solve for undetermined coefficient to find the general solution

$g(x)$	$Y_p(x)$ guess
1	$\longrightarrow A$
$5x + 7$	$\longrightarrow Ax + B$
$3x^2 - 2$	$\longrightarrow Ax^2 + Bx + C$
$x^3 - x + 1$	$\longrightarrow Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$\longrightarrow A\cos 4x + B\sin 4x$
$\cos 4x$	$\longrightarrow A\cos 4x + B\sin 4x$
e^{5x}	$\longrightarrow Ae^{5x}$
$(9x - 2)e^{5x}$	$\longrightarrow (Ax + B)e^{5x}$
$x^2 e^{5x}$	$\longrightarrow (Ax^2 + Bx + C)e^{5x}$
$e^{3x}\sin 4x$	$\longrightarrow Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
$5x^2 \sin 4x$	$\longrightarrow (Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
$xe^{3x}\cos 4x$	$\longrightarrow (Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Examples

1) Determine the general solution to $y'' - 3y' - 10y = 2e^{3x}$.

Solution

$$y'' - 3y' - 10y = 0$$

$$am^2 + bm + c = 0$$

$$m^2 - 3m - 10 = 0$$

$$m^2 + 2m - 5m - 10 = 0$$

$$m(m+2) - 5(m+2) = 0$$

$$(m-5)(m+2) = 0$$

$$m_1 = 5 \qquad m_2 = -2$$

$$y(x) = c_1 e^{5x} + c_2 e^{-2x} + Y_p$$

$$Y_p = Ae^{3x}, \quad Y_p' = 3Ae^{3x}$$

$$Y_p'' = 9Ae^{3x}$$

Now substitute Y_p, Y_p' and Y_p'' in the question, i.e.

$$9Ae^{3x} - 3(3Ae^{3x}) - 10(Ae^{3x}) = 2e^{3x}$$

$$\cancel{9Ae^{3x}} - \cancel{9Ae^{3x}} - 10Ae^{3x} = 2e^{3x}$$

$$\frac{-10A\cancel{e^{3x}}}{\cancel{e^{3x}}} = \frac{2\cancel{e^{3x}}}{\cancel{e^{3x}}}$$

$$-10A = 2 \qquad A = -\frac{1}{5}$$

Now $Y_p = -\frac{1}{5}e^{3x}$

And therefore;

$$\boxed{\therefore y(x) = c_1 e^{5x} + c_2 e^{-2x} - \frac{1}{5}e^{3x}}$$

2) Write the form of a particular soln $Y_p(x)$

$$y'' - y' - 6y = e^{3x}$$

$$am^2 + bm + c = 0$$

$$m^2 - m - 6 = 0$$

$$m^2 + 2m - 3m - 6 = 0$$

$$m(m+2) - 3(m+2) = 0$$

$$(m-3)(m+2) = 0$$

$$m_1 = 3 \qquad m_2 = -2$$

$$y(x) = c_1 e^{3x} + c_2 e^{-2x}$$

$$\boxed{\therefore Y_p(x) = Axe^{3x}}$$

3) Write the form of the particular solution $Y_p(x)$:

$$y'' + 6y' + 9y = e^{-3x}$$

$$m^2 + 6m + 9 = 0$$

$$m^2 + 3m + 3m + 9 = 0$$

$$m(m+3) + 3(m+3) = 0$$

$$(m+3)(m+3) = 0$$

$$(m+3)^2=0$$
$$m=-3$$

Now

$$y(x)=y_c+Y_p(x)$$
$$y_c=c_1e^{-3x}+c_2xe^{-3x}$$

Now check for duplication of terms of Y_p and y_c

$$\boxed{\therefore Y_p=Ax^2e^{-3x}}$$

(4) Write the form of the particular solution $Y_p(x)$.

$$9y''+y=e^{-3x}\sin\left(\frac{x}{3}\right)+x\cos\left(\frac{x}{3}\right)$$

Solution

$$9y''+y=0$$
$$9m^2+1=0$$
$$\cancel{9m^2+1}\quad 9m^2=-1$$
$$\sqrt{m^2}=\sqrt{-\frac{1}{9}}$$
$$m=\pm\frac{1}{3}i$$
$$m_1=0+\frac{1}{3}i \qquad m_2=0-\frac{1}{3}i$$
$$m_1=\alpha+\beta i \qquad m_2=\alpha-\beta i$$

~~Now~~ Recall that

$$y(x)=y_c+Y_p(x)$$
$$y_c=c_1\cos\left(\frac{x}{3}\right)+c_2\sin\left(\frac{x}{3}\right)$$

Therefore

$$\boxed{Y_p(x)=Ae^{-3x}\sin\left(\frac{x}{3}\right)+Be^{-3x}\cos\left(\frac{x}{3}\right)+x\left((Cx+D)\cos\left(\frac{x}{3}\right)+x(Ex+F)\sin\left(\frac{x}{3}\right)\right)}$$

(5) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=2e^{3x}$

Solution

$$y''-5y'+6y=2e^{3x}$$
$$am^2+bm+c=0$$
$$m^2-5m+6=0$$
$$m^2-2m-3m+6=0$$
$$m(m-2)-3(m-2)=0$$
$$(m-2)(m-3)=0$$
$$m_1=2 \qquad m_2=3$$
$$y_c=c_1e^{2x}+c_2e^{3x}$$

Now check for duplication of terms of Y_p and y_c

$$Y_p=2e^{3x}$$
$$Y_p=Axe^{3x}$$

(7)

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$Y_p' = A[e^{3x} + 3xe^{3x}]$

$Y_p'' = A[3e^{3x} + 3e^{3x} + 9xe^{3x}]$

$= A[6e^{3x} + 9xe^{3x}]$

Now substitute Y_p, Y_p' and Y_p'' in the problem

$y'' - 5y' + 6y = 2e^{3x}$

$A(6e^{3x} + 9xe^{3x}) - 5[A(e^{3x} + 3xe^{3x})]$
$+ 6Axe^{3x} = 2e^{3x}$

$e^{3x}A(6+9x) - 5Ae^{3x}(1+3x)$
$+ 6Axe^{3x} = 2e^{3x}$

Divide each term by e^{3x}

$A(6+9x) - 5A(1+3x) + 6Ax = 2$

$6A + \cancel{9Ax} - 5A - \cancel{15Ax} + \cancel{6Ax} = 2$

$A = 2$

recall that $Y_p = Axe^{3x}$

$Y_p = 2xe^{3x}$

$y(x) = y_c + Y_p$

$$\boxed{\therefore y(x) = c_1e^{2x} + c_2e^{3x} + 2xe^{3x}}$$

(6) $y'' - 6y' + 9y = 4e^{3x}$

Solution

$am^2 + bm + c = 0$

~~$m^2 - 6y$~~

$m^2 - 6m + 9 = 0$

$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$(m-3)(m-3) = 0$

$m = 3$

$y_c = c_1e^{3x} + c_2xe^{3x}$

$Y_p(x) = 4e^{3x}$

Now check for duplication of ~~terms of~~ the power; we have

$Y_p = Ax^2e^{3x}$

$Y_p' = A[x^2 \cdot 3e^{3x} + e^{3x} \cdot 2x]$

$= A[3x^2e^{3x} + 2xe^{3x}]$

$Y_p'' = A[3x^2 \cdot 3e^{3x} + 3e^{3x} \cdot 2x$
$+ 2x \cdot 3e^{3x} + 2e^{3x} \cdot 1]$

$Y_p'' = A[9x^2e^{3x} + 6xe^{3x} + 6xe^{3x}$
$+ 2e^{3x}]$

$Y_p'' = A[9x^2e^{3x} + 12xe^{3x} + 2e^{3x}]$

p.t.o

Now substitute, we have;

$$y'' - 6y' + 9y = 4e^{3x}$$

$$[9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}] - 6[3Ax^2e^{3x} + 2Axe^{3x}] + 9[Ax^2e^{3x}] = 4e^{3x}$$

$$\Rightarrow \cancel{9Ax^2e^{3x}} + \cancel{12Axe^{3x}} + 2Ae^{3x} - \cancel{18Ax^2e^{3x}} - \cancel{12Axe^{3x}} + \cancel{9Ax^2e^{3x}} = 4e^{3x}$$

$$2A\cancel{e^{3x}} = 4\cancel{e^{3x}}$$

$$\frac{\cancel{2}A}{\cancel{2}} = \frac{4}{2}$$

$$A = 2$$

Now

$$Y_p = 2x^2e^{3x}$$

And therefore

$$\boxed{y = C_1e^{3x} + C_2xe^{3x} + 2x^2e^{3x}}$$

(7) $y'' - y' - 6y = \sin x$

Solution

$$y'' - y' - 6y = 0$$

$$m^2 - m - 6 = 0$$

$$m^2 + 2m - 3m - 6 = 0$$

$$m(m+2) - 3(m+2) = 0$$

$$(m+2)(m-3) = 0$$

$$m + 2 = 0 \qquad m - 3 = 0$$

$$m_1 = -2 \qquad m_2 = 3$$

$$y = y_c + Y_p$$

$$y = C_1e^{m_1x} + C_2e^{m_2x} + Y_p$$

$$y = C_1e^{-2x} + C_2e^{3x} + Y_p$$

$$Y_p = A\sin x + B\cos x$$

$$Y_p' = A\cos x - B\sin x$$

$$Y_p'' = -A\sin x - B\cos x$$

Then substitute in the original equation

$$[-A\sin x - B\cos x] - [A\cos x - B\sin x] - 6[A\sin x + B\cos x] = \sin x$$

$$-A\sin x - B\cos x - A\cos x + B\sin x - 6A\sin x - 6B\cos x = \sin x$$

$$-7A\sin x - A\cos x - 7B\cos x + B\sin x = \sin x$$

$$-7A\sin x + B\sin x - A\cos x - 7B\cos x = \sin x$$

$$(-7A+B)\sin x - (A+7B)\cos x = \sin x$$

$$\frac{(-7A+B)\cancel{\sin x}}{\cancel{\sin x}} = \frac{\cancel{\sin x}\ 1}{\cancel{\sin x}\ 1}$$

$$-7A + B = 1 \quad --- (1)$$

Also

$$-\frac{(A+7B)\cancel{\cos x}}{\cancel{\cos x}} = \frac{0}{\cos x}$$

$$-A-7B=0 \quad \text{——— (2)}$$

By solving equation (1) and (2) simultaneously;
from equation (2)

$$A = -7B$$

substitute A in eqn (1)

$$-7(-7B)+B=1$$

$$49B+B=1$$

$$50B=1$$

$$B=\frac{1}{50}$$

Also substitute B in A

$$A=-7\left(\frac{1}{50}\right)$$

$$A=\frac{-7}{50}$$

Now

$$Y_p=-\frac{7}{50}\sin x+\frac{1}{50}\cos x$$

$$\boxed{\therefore y=C_1e^{-2x}+C_2e^{3x}-\frac{1}{50}(7\sin x-\cos x)}$$

(8) $y''-y'-6y=x^2$

Solution

$$y''-y'-6y=0$$

$$m^2-m-6=0$$

$$m^2+2m-3m-6=0$$

$$m(m+2)-3(m+2)=0$$

$$(m+2)(m-3)=0$$

$$m+2=0 \quad m-3=0$$

$$m_1=-2 \quad m_2=3$$

$$y=C_1e^{-2x}+C_2e^{3x}+Y_p$$

$$Y_p=Ax^2+Bx+C$$

$$Y_p'=2Ax+B$$

$$Y_p''=2A$$

Now substitute;

$$2A-(2Ax+B)-6(Ax^2+Bx+C)=x^2$$

$$2A-2Ax-B-6Ax^2-6Bx-6C=x^2$$

$$-6Ax^2-(2A+6B)x+2A-B-6C=x^2$$

equate the like terms

$$\frac{-6A\cancel{x^2}}{\cancel{x^2}}=\frac{\cancel{x^2}}{\cancel{x^2}}1$$

$$-6A=1$$

$$A=-\frac{1}{6}$$

Also

$$-\frac{(2A+6B)x}{x}=\frac{0}{x}$$

$$-2A+6B=0$$

$$-2A+6B=0$$
$$-2\left(-\frac{1}{6}\right)+6B=0$$
$$\frac{1}{3}+6B=0$$
$$\frac{1}{3}=-6B$$
$$1=-18B$$
$$B=-\frac{1}{18}$$

Also

$$2A-B-6C=0$$
$$2\left(-\frac{1}{6}\right)-\left(-\frac{1}{18}\right)-6C=0$$
$$-\frac{1}{3}+\frac{1}{18}=6C$$
$$\frac{-6+1}{18}=6C$$
$$-\frac{5}{18}=6C$$
$$C=-\frac{5}{108}$$

Now

$$Y_p=-\frac{1}{6}x^2-\frac{1}{18}x-\frac{5}{108}$$

$$\boxed{\therefore y=c_1e^{-2x}+c_2e^{3x}-\frac{5}{108}-\frac{1}{18}x-\frac{1}{6}x^2}$$

Mathematical Modeling Using Curve Fitting

1) Method of group average.
2) Graphical Method.
3) Method of least Square

Method of group average

Linear $\rightarrow y=ax+b$

Quadratic $\rightarrow y=ax^2+bx+c$

Example 1

x	1	2	3	4	5	6
y	2	4	7	9	12	14

Fit a straight line function $y=ax+b$ to the data above by using the method of group average. Extrapolate the value of x when $y=30$, for what value of x is $y=0$?

Solution

	x_1	y_1	x_2	y_2
	1	2	4	9
	2	4	5	12
	3	7	6	14
Σ	6	13	15	35

$x_1 = \frac{6}{3}$, $y_1 = \frac{13}{3}$

$x_1 = 2$, $y_1 = 4.3333$

$x_2 = \frac{15}{3}$, $y_2 = \frac{35}{3}$

$x_2 = 5$, $y_2 = 11.666$

$4.3333 = 2a + b$ —— (1)

$11.6667 = 5a + b$ —— (2)

By solving eqn (1) and (2) Simultaneously;

$3a = 7.3334$

$a = 2.4444$, $b = -0.5556$

$$\boxed{\therefore y = 2.4444x - 0.5556}$$

Also ~~when~~ when $y = 30$

$30 = 2.4444x - 0.5556$

$$\boxed{x = 12.5003}$$

Also when $y = 0$

$0 = 2.4444x - 0.5556$

$\frac{0.5556}{2.4444} = \frac{\cancel{2.4444x}}{\cancel{2.4444}}$

$$\boxed{x = 0.2273}$$

Example 2

Fit a curve of the form $y = ax^2 + bx + c$ for the following data by method of group average.

x	0	2	4	6	8	10
y	6	9	12	14	17	20

Solution

$y = ax^2 + bx + c$

$y_1 = ax_1^2 + bx_1 + c$

$y - y_1 = a(x^2 - x_1^2) + b(x - x_1) + (c - c)$

$y - y_1 = a(x - x_1)(x + x_1) + b(x - x_1)$

$\frac{y - y_1}{x - x_1} = a(x + x_1) + b$

let $Y = \frac{y - y_1}{x - x_1}$ and $X = x + x_1$

Now ~~the~~ the equation becomes

$Y = aX + b$

If $x_1 = 2$ and $y_1 = 9$

x	y	$Y=\frac{y-y_1}{x-x_1}$	$X=x+x_1$
0	6	3/2	2
2	9	∞	4
4	12	3/2	6
6	14	5/4	8
8	17	4/3	10
10	20	11/8	12

y_1	x_1
3/8	2
3/2	4

$\Rightarrow$ $x_1 = \frac{8}{2} = 4$

$y_1 = \frac{6}{2} = 3$

y_2	x_2
5/4	8
4/3	10
11/8	12

$x_2 = \frac{30}{3} = 10$

$\Rightarrow$ $y_2 = \frac{5}{4} + \frac{4}{3} + \frac{11}{8}$
$= 1.3194$

$y_1 = ax_1 + c \Rightarrow 3 = 4a + c$ — (1)

$y_2 = ax_2 + c \Rightarrow 1.3194 = 10a + c$ — (2)

Solving eqn (1) and (2) simultaneously

$-1.6806 = 6a$

$a = -0.2801$

$\Rightarrow$ Substitute a in eqn (1)

That is

$3 = 4(-0.2801) + c$

$c = 4.1204$

$\boxed{\therefore Y = -0.2801X + 4.1208}$

Method of Least Square

Find the straight line that best fit the following data by the least square Method.

x	1	2	3	4	5	6
y	2	4	7	9	12	14

Solution

$y = ax + b$

$\sum y = a\sum x + bn$

$\sum xy = a\sum x^2 + b\sum x$

	x	y	xy	x^2
	1	2	2	1
	2	4	8	4
	3	7	21	9
	4	9	36	16
	5	12	60	25
	6	14	84	36
$\sum$	21	48	211	91

Now we have;

$48 = 21a + 6b$ —— ①

$211 = 91a + 21b$ —— ②

By solving equation ① and ② simultaneously, we have;

$a = 2.4571$

and $b = -0.6$

$$\boxed{\therefore y = 2.4571x - 0.6}$$

Example 2

Fit a polynomial of 2^{nd} degree, using least squares method to the data points given in the following table:

X	0.0	1.0	2.0
Y	1.0	6.0	17

Also calculate the value of y at x equal to 3.

Solution

$y = ax^2 + bx + c$

$\sum y = a\sum x^2 + b\sum x + nc$ —— ①

$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$ —— ②

$\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2$ —— ③

x	y	x^2	x^3	x^4	xy	x^2y
0.0	1.0	0.0	0.0	0.0	0.0	0.0
1.0	6.0	1.0	1.0	1.0	6.0	6.0
2.0	17	4.0	8.0	16.0	34.0	68.0
$\sum$ 3.0	24.0	5.0	9.0	17.0	40.0	74.0

Now we substitute in the equations above;

$24 = 5a + 3b + 3c$

$40 = 9a + 5b + 3c$

$74 = 17a + 9b + 5c$

By solving the equations using Crammer's Rule

$$\Delta = \begin{bmatrix} 5 & 3 & 3 \\ 9 & 5 & 3 \\ 17 & 9 & 5 \end{bmatrix}$$

$|\Delta| = 5(-2) - 3(-6) + 3(-4)$
$= -4$

$$\Delta a = \begin{bmatrix} 24 & 3 & 3 \\ 40 & 5 & 3 \\ 74 & 9 & 5 \end{bmatrix}$$

$$|\Delta a| = 24(-2) - 3(-22) + 3(-10)$$
$$= -12$$

$$a = \frac{|\Delta a|}{|\Delta|} = \frac{-12}{-4} = 3$$

$$\Delta b = \begin{bmatrix} 5 & 24 & 3 \\ 9 & 40 & 3 \\ 17 & 74 & 5 \end{bmatrix}$$

$$|\Delta b| = 5(-22) - 24(-6) + 3(-14)$$
$$= -8$$

$$b = \frac{|\Delta b|}{|\Delta|} = \frac{-8}{-4} = 2$$

$$\Delta c = \begin{bmatrix} 5 & 3 & 24 \\ 9 & 5 & 40 \\ 17 & 9 & 74 \end{bmatrix}$$

$$|\Delta c| = 5(10) - 3(-14) + 24(-4)$$
$$= -4$$

$$c = \frac{|\Delta c|}{|\Delta|} = \frac{-4}{-4} = 1$$

Substitute the values of a, b and c in $y = ax^2 + bx + c$.

$$\boxed{\therefore y = 3x^2 + 2x + 1}$$

Also when $x = 3$, we have;

$$y = 3(3)^2 + 2(3) + 1$$
$$y = 3(9) + 2(3) + 1$$
$$y = 27 + 6 + 1$$

$$\boxed{\therefore y = 34}$$

"The beautiful thing about learning is that no one can take it away from you." – B.B. King

Example 3

Using method of least squares, fit a second degree polynomial to the following data.

x	1996	1997	1998	1999	2000
y	40	50	62	58	60

Solution

$$y = ax^2 + bx + c \quad \text{——— (i)}$$

$$\sum y = a\sum x^2 + b\sum x + nc \quad \text{——— (ii)}$$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x \quad \text{——— (iii)}$$

$$\sum x^2 y = a\sum x^4 + b\sum x^3 + c\sum x^2 \quad \text{——— (iv)}$$

x	y	$X = x-1998$ $X = x - \bar{x}$	X^2	X^3	X^4	Xy	X^2y
1996	40	-2	4	-8	16	-80	160
1997	50	-1	1	-1	1	-50	50
1998	62	0	0	0	0	0	0
1999	58	1	1	1	1	58	58
2000	60	2	4	8	16	120	240
9990	270	0	10	0	34	48	508

$270 = 10a + b(0) + 5c$ ——→ from (ii)

$48 = a(0) + 10b + c(0)$ ——→ from (iii)

$508 = 34a + b(0) + 10c$ ——→ from (iv)

From (iii) $48 = 10b$, $b = 4.8$

Solving (ii) and (iv) simultaneously gives $a = -2.28$ and then $c = 58.57$

Now we have

$$y = -2.28X^2 + 4.8X + 58.57$$

$$\boxed{\therefore y = -2.28(x-1998)^2 + 4.8(x-1998) + 58.57}$$

Lagrange's Interpolation Formula

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

The above expression is called Lagrange's Polynomial.

Example 1

Find the value of (i) $f(2)$ (ii) $f(2.3)$. If $f(x) = 18$, find x, given the table below

x	-1	0	3
$f(x)$	5	9	21

Solution

$x_0 = -1, \; x_1 = 0, \; x_2 = 3$

$f(x_0) = 5, \; f(x_1) = 9, \; f(x_2) = 21$

Since we have only $x_0 \sim x_2$

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Now we Substitute

$$f(x) = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} \times 5 + \frac{(x+1)(x-3)}{(0+1)(0-3)} \times 9 + \frac{(x+1)(x-0)}{(3+1)(3-0)} \times 21$$

$$f(x) = \frac{5x(x-3)}{4} - \frac{3(x+1)(x-3)}{} + \frac{7x(x+1)}{4}$$

$f(x) = 4x + 9$

Now ~~for x~~ we substitute x by its given value.

$f(2) = 4(2) + 9$

$\boxed{f(2) = 17}$

Also

$f(2.3) = 4(2.3) + 9$

$\boxed{f(2.3) = 18.2}$

And for us to find the value of x when $f(x) = 18$, we say

$f(x) = 4x + 9$

$18 = 4x + 9$

$18 - 9 = 4x$

$9 = 4x$

$\boxed{\therefore x = 9/4}$

"Education is the most powerful weapon you can use to change the world."

— BB King

Example 2

$(0,2),(1,3),(2,12),(5,147)$

Find $f(3)$ and $f'(3)$

Solution

$x_0=0,\ x_1=1,\ x_2=2,\ x_3=5$

$f(x_0)=2,\ f(x_1)=3,\ f(x_2)=12,\ f(x_3)=147$

Now we will have

$$f(x)=\frac{(x-1)(x-2)(x-5)(2)}{-10}+\frac{(x-0)(x-2)(x-5)(3)}{4}+\frac{x(x-1)(x-5)(12)}{-6}+\frac{x(x-1)(x-2)(147)}{60}$$

$$f(x)=\frac{(x-1)(x-2)(x-5)}{-5}+\frac{3x}{4}(x-2)(x-5)+2x(x-1)(x-5)+\frac{147}{60}x(x-1)(x-2).$$

If $x=3$

$$f(3)=\frac{(2)(1)(-2)}{-5}+\frac{9}{4}(1)(-2)+6(2)(-2)+\frac{147}{60}(3)(2)(1)$$

$$\boxed{\therefore f(3)=35}$$

and

$$\boxed{\therefore f'(3)=32}$$

Mathematical Modeling Using Linear Programming.

Linear Programming (L.P. also Known as Linear Optimization) is a method used in obtaining the best outcome (such as maximum profit or lowest cost) in a Mathematical Model whose requirements are represented by linear relationship.

It is a special case of Mathematical Programming (A.K.A. Mathematical Optimization)

* Objective Function
* Decision Variable
* Constraint

Example 1

Max $2x + 5y$

$x + 2y \le 16$

$5x + 3y \le 45$

$x, y \ge 0$

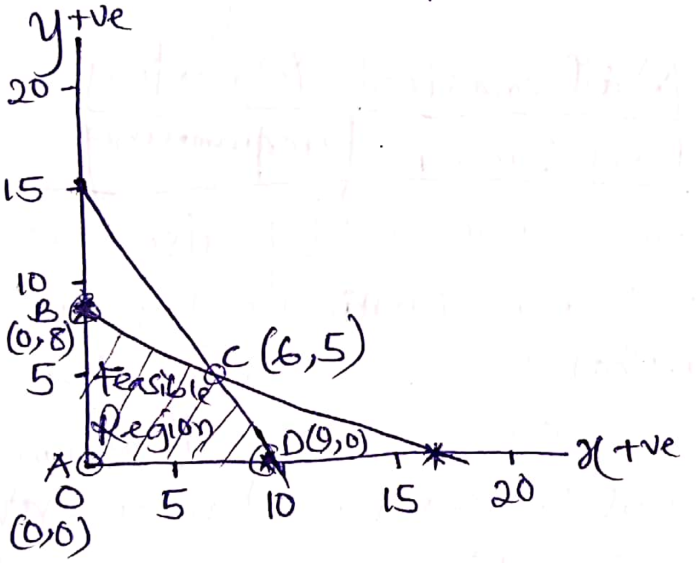
Solution

$x + 2y \le 16$

x	y
0	8
16	0

$5x + 3y \le 45$

x	y
0	15
9	0



let $x + 2y \le 16$ ——— ①

$5x + 3y \le 45$ ——— ②

Solve equation ① and ② Simultaneously to find the point C. Meanwhile we will have C(6, 5).

Point	$Z = 2x + 5y$
A(0,0)	0
B(0,8)	40
C(6,5)	37
D(9,0)	18

∴ The optimal solution $x = 0$, $y = 8$ and

$\boxed{\text{Maximum OFV} = 40}$

Example 2

Min $Z = 5x + 7y$

$x + 3y \ge 6$ ——— ①

$5x + 2y \ge 10$ ——— ②

$y \le 4$

$x, y \ge 0$

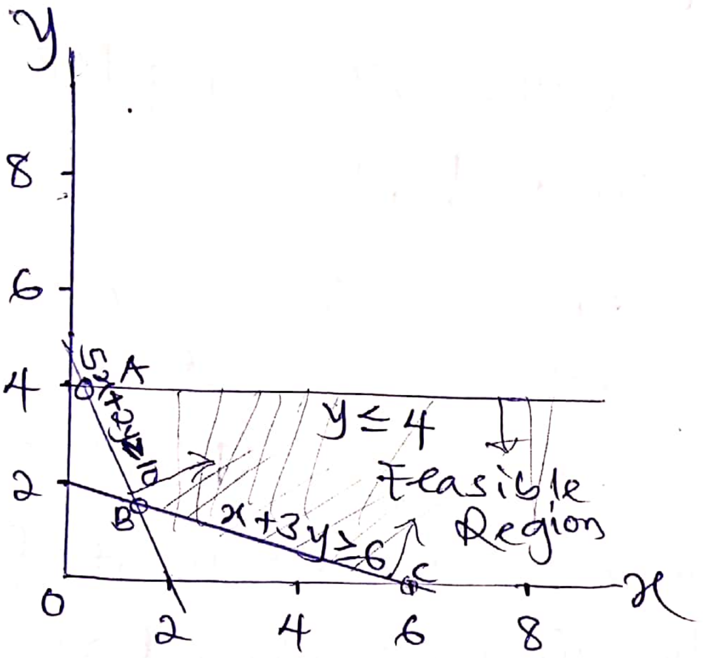
$x + 3y \ge 6$

x	y
0	2
6	0

$5x + 2y \ge 6$

x	y
0	5
2	0

$y \le 4$



Now we have point B ~~A~~ as $B(1.38, 1.54)$ by solving equation ① and ② simultaneously

therefore we say;

$$Z = 5x + 7y$$

$$Z = 5(1.38) + 7(1.54)$$

$$Z = 17.7$$

Optimal Solution $x = 1.38$ and $y = 1.54$

$$\boxed{\therefore \text{Minimum } Z = 17.7}$$

"Learning is never done without errors and defeat." – Vladimir Lenin

Laplace Transform

One Dimensional Heat Flow Equation

Example 1

Solve the following PDE using L.T.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad x > 0, \; t > 0$$

with $u(0,t) = u_0$ (constant) when $t > 0$ and $u(x,0) = 0$ when $x > 0$

Solution

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = u_0 \qquad u(x,0) = 0$$

Apply L.T. w.r.t t

$$s\bar{u}(x,s) - u(x,0) = k \frac{d^2 \bar{u}(x,s)}{dx^2}$$

$$\frac{d^2 \bar{u}}{dx^2} - \frac{s}{k} \bar{u} = 0$$

$$\bar{u}(x,s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x}$$

As $x \to \infty$, $u = 0 \Rightarrow \bar{u} = 0$

$$A = 0$$

$$\bar{u}(x,s) = Be^{-\sqrt{\frac{s}{k}}x}$$

$$\bar{u}(0,s) = B$$

$$\bar{u}(0,s) = L[u_0] = u_0 L[1] = \frac{u_0}{s}$$

$$B = \frac{u_0}{s}$$

$$\bar{u}(x,s) = u_0 \frac{e^{-\sqrt{\frac{s}{k}}x}}{s}$$

$$u(x,t) = u_0 L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{k}}x}}{s}\right]$$

$$= u_0 L^{-1}\left[\frac{e^{-\left(\frac{x}{\sqrt{k}}\right)\sqrt{s}}}{s}\right]$$

$$\boxed{\therefore u(x,t) = u_0 \, \text{erf}_c\left[\frac{x}{2\sqrt{kt}}\right]}$$

"A person who never made a mistake never tried anything new."
—Albert Einstein

Example 2

Solve the following PDE using L.T.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad x > 0, \; t > 0$$

with $u(x,0) = 0$, $u_t(x,0) = 0$ when $x > 0$ and $u(0,t) = f(t)$, $\lim_{x \to \infty} u(x,t) = 0$ when $t > 0$

Solution

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x,0) = 0, \quad u_t(x,0) = 0$$

Take L.T. on both side w.r.t. t

$$L\left[\frac{\partial^2 u}{\partial t^2}\right] = a^2 L\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$s^2 \bar{u}(x,s) - s u(x,0) - u_t(x,0) = a^2 \frac{d^2 \bar{u}}{dx^2}$$

$$\frac{d^2 \bar{u}}{dx^2} - \frac{s^2}{a^2} \bar{u} = 0$$

$$\bar{u}(x,s) = Ae^{\frac{sx}{a}} + Be^{-\frac{sx}{a}} \quad \text{---} (1)$$

$u(x,t) \to 0$ as $x \to \infty$

$\bar{u}(x,s) \to 0$ as $x \to \infty$

From (1) $\Rightarrow A = 0$

$$u(0,t) = f(t)$$

$$\bar{u}(0,s) = \int_0^\infty f(t)e^{-st}\,dt = \bar{f}(s)$$

From (1) $\Rightarrow B = \bar{f}(s)$

$$\bar{u}(x,s) = e^{-\frac{sx}{a}}\bar{f}(s)$$

$$u(x,t) = L^{-1}\left\{e^{-\frac{sx}{a}}\bar{f}(s)\right\}$$

$$= \begin{cases} f\left(t-\frac{x}{a}\right), & t > \frac{x}{a} \\ 0, & t < \frac{x}{a} \end{cases}$$

$$\boxed{u(x,t) = f\left(t-\frac{x}{a}\right)H\left(t-\frac{x}{a}\right)}$$

(in terms of Heaviside unit step function)

Table of Essential Transforms

$$L[e^{\lambda t}] = \frac{1}{s-\lambda} \leftrightarrow L^{-1}\left[\frac{1}{s-\lambda}\right] = e^{\lambda t}$$

$$L[1] = \frac{1}{s} \leftrightarrow L^{-1}\left[\frac{1}{s}\right] = 1$$

$$L[t^n] = \frac{n!}{s^{n+1}} \leftrightarrow L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$L[\cos\beta t] = \frac{s}{s^2+\beta^2} \leftrightarrow L^{-1}\left[\frac{s}{s^2+\beta^2}\right] = \cos\beta t$$

$$L[\sin\beta t] = \frac{\beta}{s^2+\beta^2} \leftrightarrow L^{-1}\left[\frac{1}{s^2+\beta^2}\right] = \frac{\sin\beta t}{\beta}$$

"ASPIRE to ACQUIRE the DESIRE that you ADMIRE, but if in the process you PERSPIRE, don't RETIRE but REFIRE to ACQUIRE the DESIRE which you ADMIRE" - Chintan Jain

let's now chill with the big boys.....

Differential Equations: Growth and Decay

Consider the statement, "The rate of change of some quantity y is directly proportional to y."

$$\frac{dy}{dt} \propto y$$

$$\frac{dy}{dt} = ky$$

If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k, then

$$y = Ce^{kt}$$

where C is the initial value of y, and k is the proportionality constant.

- When $k > 0$, Exponential growth occurs, and
- When $k < 0$, Exponential decay occurs

Example

Prove that $\frac{dy}{dt} = ky$ supports the model $y = Ce^{kt}$

Solution

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}dy = k\,dt$$

$$\int \frac{1}{y}dy = k\int dt$$

$$\ln y = kt + c$$

$$y = e^{kt+c}$$

$$y = e^{kt} \cdot e^{c}$$

$$\boxed{\therefore y = Ce^{kt}}$$

Where $e^{c} = C$

"The more that you read, the more things you will know, the more that you learn, the more places you'll go." —Dr. Seuss

Example 2

The rate of change of y is proportional to y. When $t=0$, $y=3$, and when $t=2$, $y=6$. What is the value of y when $t=3$?

Solution

$t=0$, $y=3$

$$\frac{dy}{dt} = ky$$

$$y = ce^{kt}$$

$$3 = ce^{k(0)}$$

$$3 = ce^{0}$$

$$c = 3$$

Also, when $t=2$, $y=6$

$$y = 3e^{kt}$$

$$6 = 3e^{2k}$$

$$\frac{\ln 2}{2} = \frac{2k}{2}$$

$$k = \frac{1}{2}\ln(2)$$

$k = 0.3466$ or $\ln\sqrt{2}$

And it now becomes

$$y = 3e^{\ln\sqrt{2}\,t}$$

Finally when $t=3$ $y=?$

$$y = 3e^{(\ln\sqrt{2})(3)}$$

$$\boxed{\therefore y = 8.48595}$$

Example 3

The number of bacteria in a culture is growing at a rate of $3000e^{2t/5}$ per unit of time, t. At $t=0$ the number of bacteria present was 7500. Find the number present at $t=5$.

Solution

$$\frac{dB}{dt} = 3000e^{2t/5}$$

$$\int dB = \int 3000e^{2t/5}\,dt$$

$$B = 3000e^{2t/5}\left(\frac{5}{2}\right)$$

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$$B = 7500e^{2t/5}$$

$$B(t) = 7500e^{2t/5}$$

When $t = 5$, then $B(t)$ becomes

$$B(5) = 7500e^{2(5)/5}$$

$$B(5) = 7500e^{2}$$

$$B(5) = 55417.92$$

$$\boxed{\therefore B(5) \simeq 55418}$$

Example 4

Suppose a population of insects increases according to the law of exponential growth. There were 130 insects after the third day of the experiment and 380 after the seventh day. Approximately how many insects were in the initial population?

Solution

$(3, 130)$

$$y = Ce^{kt}$$

$$130 = Ce^{k(3)}$$

$$130 = Ce^{3k}$$

$$C = \frac{130}{e^{3k}} \quad \text{(i)}$$

Also $(7, 380)$

$$380 = Ce^{k(7)}$$

$$380 = Ce^{7k} \quad \text{(ii)}$$

Substitute eqn (i) in (ii)

$$380 = \left(\frac{130}{e^{3k}}\right)e^{7k}$$

$$380 = 130e^{4k}$$

$$\ln\left(\frac{38}{13}\right) = 4k$$

$$k = \frac{1}{4}\ln\left(\frac{38}{13}\right) \simeq 0.268159$$

Now we will have;

$$y = Ce^{0.268159t}$$

$$130 = Ce^{0.268159(3)}$$

$$130 = Ce^{0.80448}$$

$$C = \frac{130}{e^{0.80448}}$$

$$\boxed{\therefore C = 58 \text{ insects initially}}$$

Newton's Law of Cooling

Newton's law of cooling states that the rate of change of temprature of the body is proportional to the difference between the body temprature at the moment and the surrounding temprature.

In other word;

Newton's law of cooling states that the temprature of an object, $T(t)$, changes at a rate proportional to the difference between the temprature of the object itself and the temprature of its surroundings, T_s.

Mathematically;

$$\frac{dT}{dt} = -k(T - T_s)$$

$$\int \frac{dT}{T - T_s} = \int -k\,dt$$

$$\ln(T - T_s) = -kt + c$$

$$T - T_s = e^{-kt + c}$$

$$T - T_s = e^{-kt} \cdot e^{c}$$

$$T - T_s = Ce^{-kt} \quad (e^{c} = C)$$

$$T(t) = T_s + Ce^{-kt}$$

Using initial temprature, $T(0) = T_0$, we get $T_0 - T_s = C$

Thus, $\boxed{\therefore T(t) = T_s + (T_0 - T_s)e^{-kt}}$

How interesting did you find the Course so far?

Okay! Let's get into Solving Test and Exams past Questions---

In case you get stuck somewhere and you want to ask a question, feel free to chat me up, via 08099549427 (Sulaiman Moriki) for better clarification.

As you understand, please guide your colleagues! So that we all succeed together.

Thank You!

Test Past Questions, Solved By S.Moriki

2020 TEST

1) If P is the pull required to lift a Load W by means of a pulley block. Find the linear law of the form $P = mW + c$ connecting P and W given the data below:

P	12	15	21	25	30	35	50
W	50	70	100	120	150	170	190

Solution

$$P = mW + c$$

$$\sum P = m\sum W + cn \quad \text{——— (i)}$$

$$\sum WP = m\sum W^2 + c\sum W \quad \text{——— (ii)}$$

	W	P	W^2	WP
	50	12	2500	600
	70	15	4900	1050
	100	21	10000	2100
	120	25	14400	3000
	150	30	22500	4500
	170	35	28900	5950
	190	50	36100	9500
$\sum$	850	188	119300	26700

From eqn (i), we will have;

$$188 = 850m + 7c \longrightarrow (iii)$$

Also from eqn (ii), we will have;

$$26700 = 119300m + 850c \longrightarrow (iv)$$

By solving eqn (iii) and (iv) simultaneously, we will have

$$m = 0.24067 \text{ and } c = -2.36767$$

$$\boxed{\therefore P = 0.24067W - 2.36767}$$

2) Solve the ODE $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 3x^2 + 6e^{2x}$ given that $y(0) = 1$, $y'(0) = 2$ using the method of undetermined coefficients.

Solution

$$y'' + y' - 6y = 0$$

$$m^2 + m - 6 = 0$$

$$m^2 - 2m + 3m - 6 = 0$$

$$m(m-2) + 3(m-2) = 0$$

$$(m-2)(m+3)=0$$

$$m-2=0 \quad m+3=0$$

$$m_1=2 \quad m_2=-3$$

$$y=C_1e^{m_1x}+C_2e^{m_2x}+Y_p$$

$$y=C_1e^{2x}+C_2e^{-3x}+Y_p$$

$$Y_p=Ax^2+Bx+C+Dxe^{2x}$$

$$Y_p'=2Ax+B+Dx\cdot 2e^{2x}+De^{2x}\cdot 1$$

$$Y_p'=2Ax+B+2Dxe^{2x}+De^{2x}$$

$$Y_p''=2A+2Dx\cdot 2e^{2x}+2De^{2x}\cdot 1+2De^{2x}$$

$$Y_p''=2A+4Dxe^{2x}+4De^{2x}$$

Now substitute Y_p, Y_p' & Y_p'' in the given question

$$(2A+4Dxe^{2x}+4De^{2x})+(2Ax+B+2Dxe^{2x}+De^{2x})$$
$$-6(Ax^2+Bx+C+Dxe^{2x})=3x^2+6e^{2x}$$

Remove the brackets, you will have:

$$2A+\cancel{4Dxe^{2x}}+4De^{2x}+2Ax+B+\cancel{2Dxe^{2x}}+De^{2x}$$
$$-6Ax^2-6Bx-6C-\cancel{6Dxe^{2x}}=3x^2+6e^{2x}$$

$$2A + 5De^{2x} + 2Ax + B - 6Ax^2 - 6Bx - 6C = 3x^2 + 6e^{2x}$$

Equate the like terms.

$$-6A\cancel{x^2} = 3\cancel{x^2}$$

$$A = -\frac{1}{2}$$

Also

$$\frac{(2A - 6B)\cancel{x}}{\cancel{x}} = \frac{0}{x}$$

$$2\left(-\frac{1}{2}\right) - 6B = 0$$

$$-1 - 6B = 0$$

$$-6B = 1$$

$$B = -\frac{1}{6}$$

$$2A + B - 6C = 0$$

$$2\left(\frac{-1}{2}\right) - \frac{1}{6} - 6C = 0$$

$$-1 - \frac{1}{6} - 6C = 0$$

$$\frac{-6 - 1 - 36C}{6} = 0$$

In case you misunderstand something, feel free to call me via 08099549427 (Moriki) or chat me up for better clarification.

$$-7-36C=0$$

$$\frac{\cancel{-36}C}{\cancel{-36}}=\frac{7}{-36}$$

$$C=-\frac{7}{36}$$

Also

$$5D\cancel{e^{2x}}=6\cancel{e^{2x}}$$

$$D=\frac{6}{5}$$

Now substitute A, B, C and D in the general solution;

$$y=C_1e^{2x}+C_2e^{-3x}-\frac{1}{2}x^2-\frac{1}{6}x-\frac{7}{36}+\frac{6}{5}xe^{2x}$$

Now by taking the initial value problem as $y(0)=1$, we have;

$$y(0)=C_1e^{2(0)}+C_2e^{-3(0)}-\frac{1}{2}(0)^2-\frac{1}{6}(0)-\frac{7}{36}+\frac{6}{5}(0)e^{2(0)}$$

$$1=C_1+C_2-0.1944$$

$$1.1944=C_1+C_2 \text{ ——— } (1)$$

Also for $y'(0)=2$;

$$y'=2C_1e^{2x}-3C_2e^{-3x}-x-\frac{1}{6}+\frac{6}{5}x.2e^{2x}+\frac{6}{5}e^{2x}.1$$

$$y'(0)=2C_1e^{2(0)}-3C_2e^{-3(0)}-0-0.1667+0+1.2$$

$$0.9667=2C_1-3C_2 \text{ ——— } (2)$$

By solving eqn 1 and 2 simultaneously, we will have ;

$C_1 = 0.90998$ and $C_2 = 0.28442$

Therefore our General Equation now becomes

$$\boxed{y = 0.90998e^{2x} + 0.28442e^{-3x} - \frac{1}{2}x^2 - \frac{1}{6}x - \frac{7}{36} + \frac{6}{5}xe^{2x}}$$

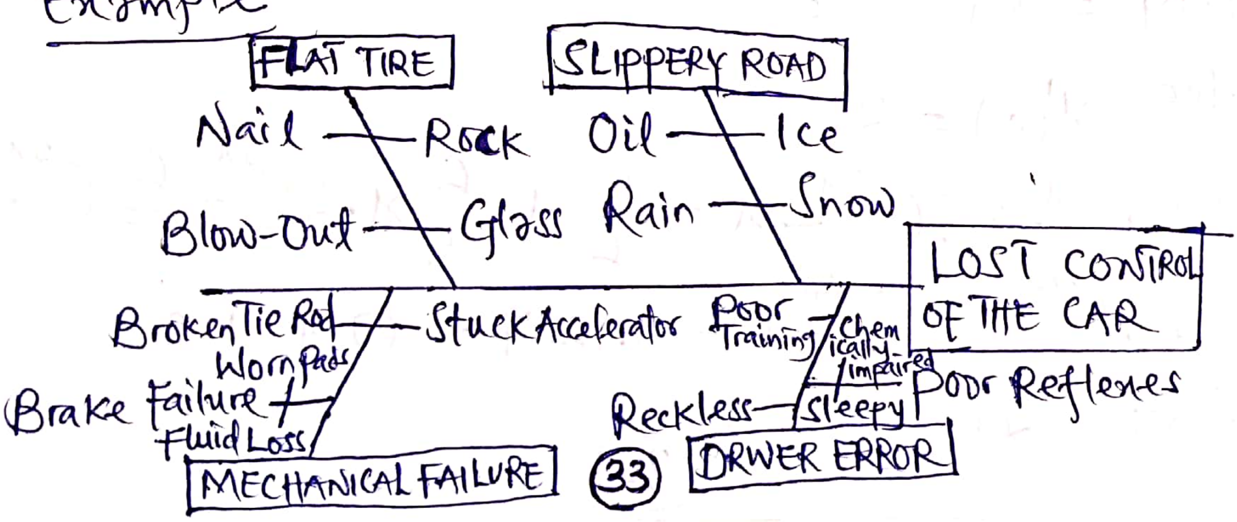
2019 TEST

1) Discuss the cause and effect diagram with examples.

Answer

A cause-effect diagram is a visual tool used to logically organize possible causes for a specific problem or effect by graphically displaying them in increasing detail, suggesting causal relationships among theories. A popular type is also referred to as a fishbone or Ishikawa diagram.

Example



② Obtain the particular solution of the IVP $y'' - 4y' + 4y = 0$, at $y(0) = 12$ and $y'(0) = -3$.

Solution

$$y'' - 4y' + 4y = 0$$

Let $y = e^{mx}$, $y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 4me^{mx} + 4e^{mx} = 0$$

$$\frac{\cancel{e^{mx}}(m^2 - 4m + 4)}{\cancel{e^{mx}}} = \frac{0}{e^{mx}}$$

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$m - 2 = 0 \qquad m - 2 = 0$$

$$m_1 = 2 \qquad m_2 = 2$$

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

At $y(0) = 12$

$$y(0) = C_1 e^{2(0)} + C_2 (0) e^{2(0)}$$

$$12 = C_1$$

Also at $y'(0) = -3$

$$y' = 2C_1 e^{2x} + C_2 x \cdot 2e^{2x} + C_2 e^{2x} \cdot 1$$

$$y'(0) = 2C_1 e^{2(0)} + 2C_2 (0) e^{2(0)} + C_2 e^{2(0)}$$

$$-3 = 2C_1 + C_2 \quad \text{--- (1)}$$

Substitute C_1 in eqn (1)

$$-3 = 2(12) + C_2$$

$$-3 = 24 + C_2$$

$$C_2 = -27$$

$$\boxed{\therefore y = 12e^{2x} - 27xe^{2x}}$$

(3) Fit a curve of the form $y = ax + b$ to the data below using method of group average
$\{(0,6), (2,9), (4,12), (6,15), (8,18), (10,21)\}$.

Solution

Check page (11) and (12) for details on how to go about it, it is very simple!

4 A slow economy caused a company's annual revenue to drop from \$530,000 in 2012 to \$380,000 in 2014. If the revenue is following an exponential pattern of decline. What is the respective expected revenue in 2016 and 2018?

Solution

At initial year, we have;

$t=0$ and $y=530000$

$$y=ce^{kt}$$

$$530000=ce^{k\times 0}$$

$$c=530000$$

In 2014, we have;

$t=2$ $y=380000$

$$y=ce^{kt}$$

$$\frac{380000}{530000}=\frac{\cancel{530000}e^{2k}}{\cancel{530000}}$$

$$e^{2k}=0.71698$$

$$2k=\ln(0.71698)$$

$$2k=-0.3327$$

$$k=-0.16635$$

In 2016, we will have;

$t=4$

$$y=ce^{kt}$$

$$y=530000e^{(-0.16635)4}$$

$$y=530000e^{-0.6654}$$

$$y=272,455.9655$$

$$\boxed{\therefore y\simeq 272,456}$$

Also in 2018

$t=6$

$$y=ce^{kt}$$

$$y=530000e^{(-0.16635)6}$$

$$y=530000e^{(-0.9981)}$$

$$y=195,346.9106$$

$$\boxed{\therefore y\simeq 195,347}$$

Solution

x_1	y_1
0	6
2	9
4	12
Σ 6	27

x_2	y_2
6	15
8	18
10	21
Σ 24	54

Now by ~~findin~~ finding the mean of each above;

$x_1 = \frac{6}{3}$ $\quad y_1 = \frac{27}{3}$ $\quad x_2 = \frac{24}{3}$ $\quad y_2 = \frac{54}{3}$

$x_1 = 2$ $\quad y_1 = 9$ $\quad x_2 = 8$ $\quad y_2 = 18$

$$y_1 = ax_1 + b \quad \text{——— (1)}$$
$$y_2 = ax_2 + b \quad \text{——— (2)}$$

Now substitute y_1, x_1, y_2 and x_2 in eqn (1) and (2) respectively;

$$9 = 2a + b \quad \text{——— (3)}$$
$$18 = 8a + b \quad \text{——— (4)}$$

By solving eqn (3) and (4) simultaneously we will have $a = \frac{3}{2}$ and $b = 6$

$$\boxed{\therefore y = 1.5x + 6}$$

Exams Past Question 2020

Answer Any Four Questions - - -

1a) Use Laplace transform technique to find the solution of PDE $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial y^2}$; $t \le 0: u=0$ for all values y; $t > 0: \begin{cases} u=1 \text{ at } y=0 \\ u \to 0 \text{ as } y \to \infty \end{cases}$

Solution

Check page 20 to get details on how to go about it.

1b) Define the error function and Complementary error function.

Answer

Error function is a special function that occurs in probabily, statistics, and partial differential equations describing diffusion. It is defined as: $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

Complementary Error Function is defined as the area under the two tails of a zero-mean. It is defined as:

$$erfc = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = e^{-x^2} erfcx(x).$$

2a) A slow economy caused a company's annual revenue to frop from \$530,000 in 2012 to \$380,000 in 2014. If the revenue is following an exponential pattern of decline. What is the respective expected revenue in 2016 and 2018?

Solution

Check page 36 - - - - !

2b) Use the Lagranges interpolation approach to fit a polynomial for the following data set; $(x,y) = \{(-1,5), (0,9), (3,21)\}$. For what value of x is $y=0$.

Solution

$x_0 = -1, x_1 = 0, x_2 = 3 \qquad y_0 = 5, \ y_1 = 9, \ y_2 = 21$

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

$$y = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} \times 5 + \frac{(x-(-1))(x-3)}{(0-(-1))(0-3)} \times 9 + \frac{(x-(-1))(x-0)}{(3-(-1))(3-0)} \times 21$$

$$y = \frac{5}{4}(x^2 - 3x) - 3(x^2 - 2x - 3) + \frac{21}{12}(x^2 + x)$$

$$y = \frac{\cancel{15x^2} - 45x - \cancel{36x^2} + 72x + 118 + \cancel{21x^2} + 21x}{12}$$

$$y = \frac{48x + 118}{12}$$

$$\boxed{\therefore y = 4x + 9.83}$$

For what value of x is $y = 0$?

Now it becomes

$$0 = 4x + 9.83$$

$$\frac{\cancel{4}x}{\cancel{4}} = \frac{-9.83}{4}$$

$$\boxed{\therefore x = -2.4575}$$

3a) A colony of bacteria is growing exponentially. At time $t = 2$ it has 10 bacteria in it, and at time $t = 4$ it has 2000. At what time will it have 100,000 bacteria?

Solution

$$y = Ce^{kt}$$

At $t = 2$, $y = 10$

$$10 = Ce^{2k} \quad (1)$$

Also at $y = 2000$, $t = 4$

$$2000 = Ce^{4k} \quad (2)$$

divide eqn (2) by (1)

$$\frac{200\cancel{0}}{1\cancel{0}} = \frac{Ce^{4k}}{Ce^{2k}}$$

$$200 = e^{2k}$$

$$\ln(200) = 2k$$

$$5.2983 = 2K$$

$$K = 2.64915$$

$K = 2.64915$

Now by taking eqn ①

$10 = Ce^{2\times 2.64915}$

$10 = Ce^{5.2983}$

$\frac{10}{199.9965} = \frac{\cancel{199.9965}C}{\cancel{199.9965}}$

$C = 0.05$

Now at $y = 100000$, $K = 2.64915$, $C = 0.05$, $t = ?$

$y = Ce^{kt}$

$\frac{100000}{0.05} = \frac{\cancel{0.05}e^{2.64915t}}{\cancel{0.05}}$

$2000000 = e^{2.64915t}$

$\ln(2000000) = 2.64915t$

$\frac{14.50866}{2.64915} = \frac{\cancel{2.64915}t}{\cancel{2.64915}}$

$\boxed{\therefore t = 5.4767}$

3b) Obtain the general solution to the ODE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 2\cos 2x$ using method of undetermined coefficients.

Solution

$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 2\cos 2x$

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$

$m^2e^{mx} - 5me^{mx} + 4e^{mx} = 0$

$\frac{\cancel{e^{mx}}(m^2 - 5m + 4)}{\cancel{e^{mx}}} = \frac{0}{e^{mx}}$

$m^2 - 5m + 4 = 0$

$m^2 - m - 4m + 4 = 0$

$m(m-1) - 4(m-1) = 0$

$(m-1)(m-4) = 0$

$m = 1 \quad m = 4$

$y = C_1e^{m_1x} + C_2e^{m_2x} + Y_p$

$y = C_1e^{x} + C_2e^{4x} + Y_p$

$Y_p = A\cos 2x + B\sin 2x$

$Y_p' = -2A\sin 2x + 2B\cos 2x$

$Y_p'' = -4A\cos 2x - 4B\sin 2x$

Then substitute;

$(-4A\cos 2x - 4B\sin 2x) - 5(-2A\sin 2x + 2B\cos 2x) + 4(A\cos 2x + B\sin 2x) = 2\cos 2x$

$-\cancel{4A\cos 2x} - \cancel{4B\sin 2x} + 10A\sin 2x - 10B\cos 2x + \cancel{4A\cos 2x} + \cancel{4B\sin 2x} = 2\cos 2x$

$$\frac{\cancel{10}A\cancel{\sin 2x}}{\cancel{10\sin 2x}} = \frac{0}{10\sin 2x}$$

$A = 0$

Also

$$\frac{\cancel{-10}B\cancel{\cos 2x}}{\cancel{-10\cos 2x}} = \frac{2\cancel{\cos 2x}}{-10\cancel{\cos 2x}}$$

$B = -\frac{1}{5}$

$Y_p = 0\cos 2x - \frac{1}{5}\sin 2x$

$Y_p = -\frac{1}{5}\sin 2x$

$$\boxed{\therefore y = C_1e^{x} + C_2e^{4x} - \frac{1}{5}\sin 2x}$$

4) A farmer has 240 acres of land. He gains a profit of \$40/acre if he grows corn and \$30/acre if he grows oats. The total labor hours he has at hand is 320 hours. Corn takes 2 hours per acre of labor to grow and oat requires 1 hour per acre of labor to grow. Using the linear programming approach, how many acres of each he should plant in order to maximize profit?

Solution

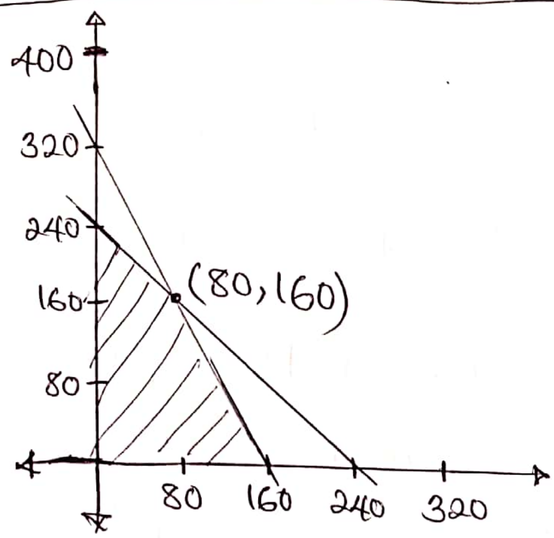
let,
x represents acres of corn
y represents acres of oat

Objective Function:
Maximize Profit $= 40x + 30y$

where,
$40x$ = total income from corn
$30y$ = total income from oat

Land = 240 acres
Labor hours = 320 hours
Corn $\longrightarrow$ 2 hours of labor
Oat $\longrightarrow$ 1 hour of labor.

Constraints into Inequalities:
$x + y \le 240$
$2x + y \le 320$
$x \ge 0,\ y \ge 0$



$\therefore$ Maximum profit will be gained if 80 acres of corn and 160 acres of oat is grown.

(5) Solve the differential equation below using the method of undetermined coefficients;

$$y'' + y' - 6y = 3x^2 + 6e^{2x} \text{ given that } y(0) = 1,\ y'(0) = 2.$$

Solution

Check page (21) — — !

(6) Certain experimental values of x and y are given below

x	0.0	2.0	5.0	7.0
y	-1.0	5.0	12.0	20.0

If $y = b_0 + b_1 x + b_2 x^2$, find the best fit values of b_0 and b_1 and b_2, by using least squares method. Also calculate the value of y at $x = 10$.

Solution

$$y = b_0 + b_1 x + b_2 x^2$$

$$\sum y = nb_0 + b_1 \sum x + b_2 \sum x^2 \text{ ——— (i)}$$

$$\sum xy = b_0 \sum x + b_1 \sum x^2 + b_2 \sum x^3 \text{ ——— (2)}$$

$$\sum x^2 y = b_0 \sum x^2 + b_1 \sum x^3 + b_2 \sum x^4 \text{ ——— (3)}$$

	x	y	x^2	x^3	x^4	xy	x^2y
	0.0	-1.0	0.0	0.0	0.0	0.0	0.0
	2.0	5.0	4.0	8.0	16.0	10.0	20.0
	5.0	12.0	25.0	125.0	625.0	60.0	300.0
	7.0	20.0	49.0	343.0	2401.0	140.0	980.0
$\sum$	14.0	36.0	78.0	476.0	3042.0	210.0	1300.0

Now substitute the values in the above equations

$$36 = 4b_0 + 14b_1 + 78b_2$$
$$210 = 14b_0 + 78b_1 + 476b_2$$
$$1300 = 78b_0 + 476b_1 + 3042b_2$$

Now by solving the above equations using Crammer's Rule, we have;

$$\Delta = \begin{bmatrix} 4 & 14 & 78 \\ 14 & 78 & 476 \\ 78 & 476 & 3042 \end{bmatrix}$$

$$|\Delta| = 42800 - 76440 + 45240$$
$$= 11600$$

$$\Delta b_0 = \begin{bmatrix} 36 & 14 & 78 \\ 210 & 78 & 476 \\ 1300 & 476 & 3042 \end{bmatrix}$$

$$|\Delta b_0| = 385200 - 280280 - 112320$$
$$= -7400$$

$$b_0 = \frac{|\Delta b_0|}{|\Delta|} = \frac{-7400}{11600}$$

$$b_0 = -0.6379$$

$$\Delta b_1 = \begin{bmatrix} 4 & 36 & 78 \\ 14 & 210 & 476 \\ 78 & 1300 & 3042 \end{bmatrix}$$

$$|\Delta b_1| = 80080 - 196560 + 141960$$
$$= 25480$$

$$b_1 = \frac{|\Delta b_1|}{|\Delta|} = \frac{25480}{11600} = 2.1966$$

$$\Delta b_2 = \begin{bmatrix} 4 & 14 & 36 \\ 14 & 78 & 210 \\ 78 & 476 & 1300 \end{bmatrix}$$

$$|\Delta b_2| = 5760 - 25480 + 20880$$
$$= 1160$$

$$b_2 = \frac{|\Delta b_2|}{|\Delta|} = \frac{1160}{11600} = 0.1$$

$$\therefore y = -0.6379 + 2.1966x + 0.1x^2$$

Also when $x = 10$

$$y = -0.6379 + 2.1966 \times 10 + 0.1 \times 10^2$$
$$y = -0.6379 + 21.966 + 10$$

$$\boxed{\therefore y = 31.3281}$$

Exams past Question 2018

Answer Any Four---

1) The rate of increase of the Population of bacteria in culture is proportional to the number of bacteria at that instant. Assume the initial count of bacteria is 12,00 and after one hour the count is 1400. Find (i) the number of bacteria present immediately after 4 hours. (ii) the time laps before the number of bacteria reaches 6000. (15 marks)

Solution

$y_0 = 1200 \qquad t_0 = 0$

$y_0 = ce^{kt_0}$

$1200 = ce^{k\times 0}$

$1200 = c$

$y = 1400 \qquad t = 1hr$

$\frac{1400}{1200} = \frac{\cancel{1200}e^k}{\cancel{1200}}$

$e^k = 1.16667$

$k = \ln(1.16667)$

$k = 0.15415$

Now when $t = 4$

i) $y = 1200e^{0.15415\times 4}$

$y = 1200e^{0.6166}$

$\boxed{y = 2223}$

ii) when $y = 6000$

$6000 = 1200e^{0.15415t}$

$e^{0.15415t} = 5$

$\frac{0.15415t}{0.15415} = \frac{\ln(5)}{0.15415}$

$t = 10.44$

$\boxed{t \simeq 10 \text{ hours}}$

2) Certain experiment values of x and y are given below:

x	0.0	2.0	5.0	7.0
y	-1.0	5.0	12.0	20.0

If $y = a_0 + a_1x$, find the best fit values of a_0 and a_1, by using least squares method. Also calculate y at $x = 5$. (15 marks)

Solution

$y = a_0 + a_1 x$

$\sum y = 4a_0 + a_1 \sum x$ --- (1)

$\sum xy = a_0 \sum x + a_1 \sum x^2$ —— (2)

x	y	x^2	xy
0.0	-1.0	0.0	0.0
2.0	5.0	4.0	10.0
5.0	12.0	25.0	60.0
7.0	20.0	49.0	140.0
$\sum$ 14.0	36.0	78.0	210.0

Now substitute;

$36 = 4a_0 + 14a_1$ —— (3)

$210 = 14a_0 + 78a_1$ —— (4)

By solving equation (3) and (4) simultaneously

We have:

$a_0 = -1.1381$

$a_1 = 2.8966$

$\therefore y = -1.1381 + 2.8966x$

$$\boxed{\therefore y = 2.8966x - 1.1381}$$

When $x = 5$

$y = 2.8966 \times 5 - 1.1381$

$$\boxed{\therefore y = 13.3449}$$

"Never let the fear of striking out stop you from playing the game."
— Babe Ruth

3) Determine the constant a and b by the method of least squares such that $y = a\exp(bx)$ fit the following data: (15 marks)

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

Solution

$$y = ae^{bx}$$

$$\ln y = \ln a + \ln e^{bx}$$

$$\ln y = \ln a + bx$$

$$Y = A + bx$$

$$\sum Y = 5A + b\sum x \quad \text{———} (1)$$

$$\sum xY = A\sum x + b\sum x^2 \quad \text{———} (2)$$

x	y	x^2	$Y(\ln y)$	xY
2	4.077	4	1.405	2.810
4	11.084	16	2.406	9.624
6	30.128	36	3.405	20.430
8	81.897	64	4.405	35.240
10	222.62	100	5.405	54.050
Σ 30	349.806	220	17.026	122.154

Substitute the value in equation (1) and (2) respectively;

$$17 = 5A + 30b \quad \text{———} (3)$$

$$122 = 30A + 220 \quad \text{———} (4)$$

By solving equation (3) and (4), we have;

$$A = 0.4 \text{ and } b = 0.5$$

Recall that

$$A = \ln a$$

$$a = e^A$$

$$a = e^{0.4}$$

$$a = 1.4918$$

From the original eqn ($y = ae^{bx}$), substitute the values of a and b.

$$\boxed{\therefore\ y = (1.4918)e^{0.5x}}$$

"Success is the sum of small efforts, repeated."

—R Collier

4) A manufacture produces two types of models M_1 and M_2. Each M_1 model requires 4 hours of grindings and 2 hours of polishing; whereas each M_2 model requires 2 hours of grindings and 5 hours of polishing. The manufacture has two grinders and three polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on an M_1 model is three naira and on an M_2 model is 4 naira. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week? Use graphical method of linear programming. (15 marks)

Solution

	M_1	M_2

Grinding hrs $4x + 2y \le 2 \times 40$

Polishing hrs $2x + 5y \le 3 \times 60$

$$Z = 3x + 4y$$

$$4x + 2y \le 80$$

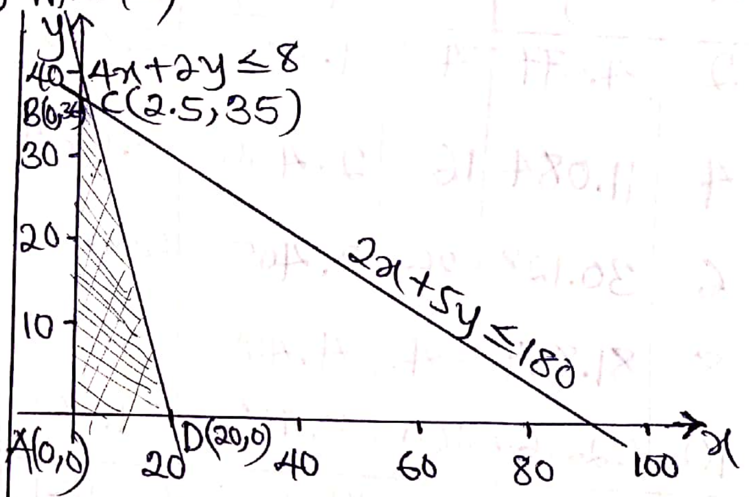
$$2x + 5y \le 180$$

$$x \ge 0;\ y \ge 0;$$

for $4x+2y=80$, for $2x+5y=180$

x	y
0	40
20	0

x	y
0	36
90	0



Point	$Z = 3x+4y$
A(0,0)	0
B(0,36)	144
C(2.5,35)	147.5
D(20,0)	60

∴ The maximum profit is 147.5 when the production capacity of M_1 is 2.5 and that of M_2 is 35

(5) Solve the differential equation by using method of undetermined coefficient.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 3xe^{x}\sin(2x) \quad \boxed{15\text{ mrks}}$$

Solution

Check page (5) for details on how to go about it.

(6) A beaker of water initially at 100°c is allowed to cool in a room maintained temprature 10°C. After 2 minutes, the water temprature is 60°C. Find the temprature as function of time. Also evaluate: (a) The water temprature after 4 minutes. (b) The time taken for water to reach 40°C. [15mrks]

Solution

$y_0 = 100°c \quad t_0 = 0$

$y_0 = 10 + ce^{-kt_0}$

$100 = 10 + ce^{-k\times 0}$

$c = 90$

At $t_1 = 2 \quad y_1 = 60°c$

$y_1 = 10 + ce^{-kt_1}$

$60 = 10 + 90e^{-2k}$

$50 = 90e^{-2k}$

$e^{-2k} = 0.55556$

$-2k = -0.58778$

$k = +0.29389$

Let's check page (51) for better look a like solution

(a) $y_2 = ?$ at $t = 4$ minutes

$y = 10 + 90e^{-0.29389\times 4}$

$y = 37.778$

$\boxed{y \simeq 38°}$

(b) when $y = 40°c$; $t = ?$

$40 = 10 + 90e^{-0.29389\times t}$

$\frac{30}{90} = e^{-0.29389t}$

$-0.29389t = -1.09862$

$t = \frac{-1.09862}{-0.29389}$

$t = 3.7382$

$\boxed{t \simeq 4 \text{ minutes}}$

Past Questions 2017
Answer Any Four Questions---

1a) Define convex region. (5 mrks)

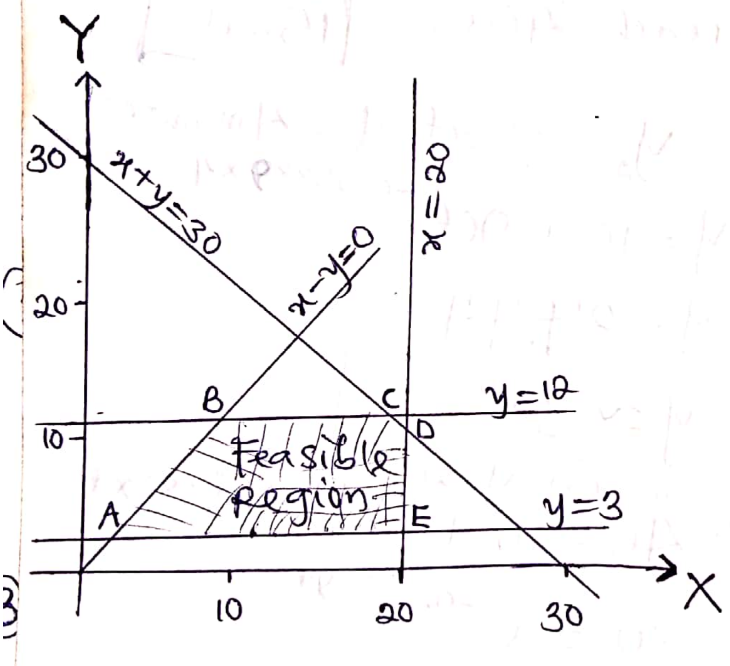
Answer

A convex region is defined as a part of a plane such that a section connecting any two of its points is entirely contained within that part of the plane.

1b) Find the maximum value of

$Z = 2x + 3y$ subject to the constraints: $x + y \leq 30$, $y \geq 3$, $0 \leq y \leq 12$, $x - y \geq 0$, $0 \leq x \leq 20$. [10 mrks]

Solution



Point	$Z = 2x + 3y$
A(3,3)	→ 15
B(12,12)	→ 60
C(18,12)	→ 72
D(20,10)	→ 70
E(20,3)	→ 49

$\therefore$ The maximum value of the Objective function is 72 at point C(18,12).

"You are braver than you believe, stronger than you seem and smarter than you think." - A.A Milne

2) The rate of increase of the population of bacteria in culture is proportional to the number of bacteria at that instant. Assume the initial count of bacteria is 1000 and after one hour the count is 1200. Find (i) the number of bacteria present immediately after 5 hours. (ii) the time laps before the number of bacteria reaches 4000. [15 mrks].

Solution

$y = 1000 \quad t = 0$

$y = ce^{kt}$

$1000 = ce^{k \times 0}$

$c = 1000$

$y = 1200 \quad t = 1$

$y = ce^{kt}$

$1200 = 1000e^{k \times 1}$

$e^{k} = 1.2$

$k = 0.18232$

(i) $y = ce^{kt}$

$y = 1000e^{0.18232 \times 5}$

$y = 1000 \times e^{0.91161}$

$y = 1000 \times 2.48833$

$y = 2488.33$

$\boxed{\therefore y \simeq 2488}$

(ii) $y = 4000 \quad t = ?$

$y = ce^{kt}$

$4000 = 1000e^{0.18232 \times t}$

$e^{0.18232t} = 4$

$0.18232t = \ln(4)$

$t = \frac{1.38629}{0.18232}$

$t = 7.60363$

$\boxed{\therefore t \simeq 8 \text{ hours}}$

3) Certain experimental values of x and y are given below:

X	0.0	2.0	5.0	7.0
Y	-1.0	5.0	12.0	20.0

If $y = a_0 + a_1 x$, find the best fit values of a_0 and a_1, by using least squares method. Also calculate y at $x = 5$.
[15 mrks]

Solution
Check page 44 and 45 ---!

4) Fit a polynomial of 2nd degree, using least squares method to the data points given in the following table:

X	0.0	1.0	2.0
Y	1.0	6.0	17

Also calculate the value of y at x equal to 3.

[15 marks]

Solution

Check page (14) ---- !

5) A beaker of water initially at 100°C is allowed to cool in a room maintained at temprature 20°C. After 2 minutes, the water temprature is 80°C. Find the temprature as function of time. Also evaluate: (a) the water temprature after 4 minutes. (b) The time taken for water to reach 50°C.

[15 marks]

Solution

$$\frac{dy}{dt} \propto [y(t) - 20]$$

$$\frac{dy}{dt} = -k[y(t) - 20]$$

$$\frac{dy}{[y(t)-20]} = -k\,dt$$

$$\int \frac{1}{y(t)-20} \cdot dy = -k\int dt$$

$$\ln[y(t)-20] = -kt + c$$

$$y(t) - 20 = e^{-kt+c}$$

$$y(t) = 20 + e^{-kt} \cdot e^{c}$$

$$\boxed{y(t) = 20 + Ce^{-kt}}$$

Now $y(t) = 100$ $t = 0$

$$100 = 20 + Ce^{-k\times 0}$$

$$100 - 20 = C \times 1$$

$$C = 80$$

③

Also $y(t) = 80°C$ $t = 2$ minutes

$$80 = 20 + 80e^{-2k}$$

$$\frac{60}{80} = e^{-2k}$$

$$-2k = -0.28768$$

$$k = +0.14384 = 0.14384$$

ⓐ $y(t) = ?$ when $t = 4$ minutes

$$y(t) = 20 + 80e^{-0.14384 \times 4}$$

$$y(t) = 20 + 80e^{-0.57536}$$

$$\boxed{\therefore y(t) = 65°C}$$

ⓑ $y(t) = 50°C$ $t = ?$

$$y(t) = 20 + ce^{-kt}$$

$$50 = 20 + 80e^{-0.14384 \times t}$$

$$30 = 80e^{-0.14384t}$$

$$e^{-0.14384t} = 0.375$$

$$-0.14384t = -0.98083$$

$$t = 6.81889$$

$$\boxed{\therefore t = 6.8 \text{ minutes}}$$

⑥ Solve the differential equation by using method of undetermined coefficient.

ⓐ $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

ⓑ $\frac{d^2y}{dx^2} + y = x\exp(x)$

[15 mrks]

Solution

ⓐ $m^2 - 5m + 6 = 0$

$$m^2 - 2m - 3m + 6 = 0$$

$$m(m-2) - 3(m-2) = 0$$

$$(m-2)(m-3) = 0$$

$$m_1 = 2 \text{ and } m_2 = 3$$

$$y = C_1e^{m_1x} + C_2e^{m_2x} + Y_p$$

$$y = C_1e^{2x} + C_2e^{3x} + Y_p$$

But

$$Y_p = Ax^2 + Bx + C$$

$$Y_p' = 2Ax + B$$

$$Y_p'' = 2A$$

Now substitute in the original equation;

$$2A - 5(2Ax + B) + 6(Ax^2 + Bx + C) = x^2$$

$$2A - 10Ax - 5B + 6Ax^2 + 6Bx + 6C = x^2$$

Equate the like terms;

$$\frac{6A\cancel{x^2}}{6\cancel{x^2}} = \frac{\cancel{x^2}}{6\cancel{x^2}}$$

$$A = \frac{1}{6}$$

$$\frac{(-10A + 6B)\cancel{x}}{\cancel{x}} = \frac{0}{x}$$

$$-10A + 6B = 0$$

$$-10 \times \frac{1}{6} + 6B = 0$$

$$-\frac{5}{3} + 6B = 0$$

$$6B = \frac{5}{3}$$

$$B = \frac{5}{18}$$

Also

$$2A - 5B + 6C = 0$$

$$2 \times \frac{1}{6} - 5 \times \frac{5}{18} + 6C = 0$$

$$\frac{1}{3} - \frac{25}{18} + 6C = 0$$

$$6C = \frac{25}{18} - \frac{1}{3}$$

$$6C = \frac{25 - 6}{18}$$

$$6C = \frac{19}{18}$$

$$C = \frac{19}{108}$$

$$Y_p = \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}$$

$$\therefore y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}$$

"A man's mind stretched by new ideas, may never return to its original dimensions." - Oliver Wendell Holmes Jr.

(b) $m^2 + 1 = 0$

$m^2 = -1$

$\sqrt{m^2} = \pm\sqrt{-1}$

$m = \pm i$

$m_1 = 0 + 1i \quad m_2 = 0 - 1i$

$\alpha = 0, \beta = 1$

$y = e^{\alpha x}[C_1 \cos(\beta x) + C_2 \sin(\beta x)] + Y_p$

$y = e^{0 \times x}[C_1 \cos x + C_2 \sin x] + Y_p$

$y = C_1 \cos x + C_2 \sin x + Y_p$

$Y_p = (Ax + B)e^x$

$Y_p' = (Ax + B)e^x + Ae^x$

$Y_p'' = (Ax + B)e^x + Ae^x + Ae^x$

Now substitute in the original equation;

$(Ax + B)e^x + 2Ae^x + (Ax + B)e^x = xe^x$

$2(Ax + B)\cancel{e^x} + 2A\cancel{e^x} = x\cancel{e^x}$

$2Ax + 2B + 2A = x$

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Equate the like terms;

$\frac{\cancel{2Ax}}{\cancel{2x}} = \frac{x}{2x}$

$A = \frac{1}{2}$

$2A + 2B = 0$

$2 \times \frac{1}{2} + 2B = 0$

$2B = -1$

$B = -\frac{1}{2}$

$Y_p = \left(\frac{1}{2}x - \frac{1}{2}\right)e^x$

$Y_p = \frac{1}{2}(x - 1)e^x$

$$\boxed{\therefore y = C_1 \cos x + C_2 \sin x + \frac{1}{2}(x-1)e^x}$$

End of the course!